Part 3
Truncation Errors

Key Concepts

- Truncation errors
- Taylor's Series
  - To approximate functions
  - To estimate truncation errors
- Estimating truncation errors using other methods
  - Alternating Series, Geometry series, Integration

Introduction

How do we calculate

$$\sin(x),\ \cos(x),\ e^x,\ x^y,\ \sqrt{x},\ \log(x),\ \ldots$$

on a computer using only +, -, x, ÷?

One possible way is via summation of infinite series. e.g.,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$$

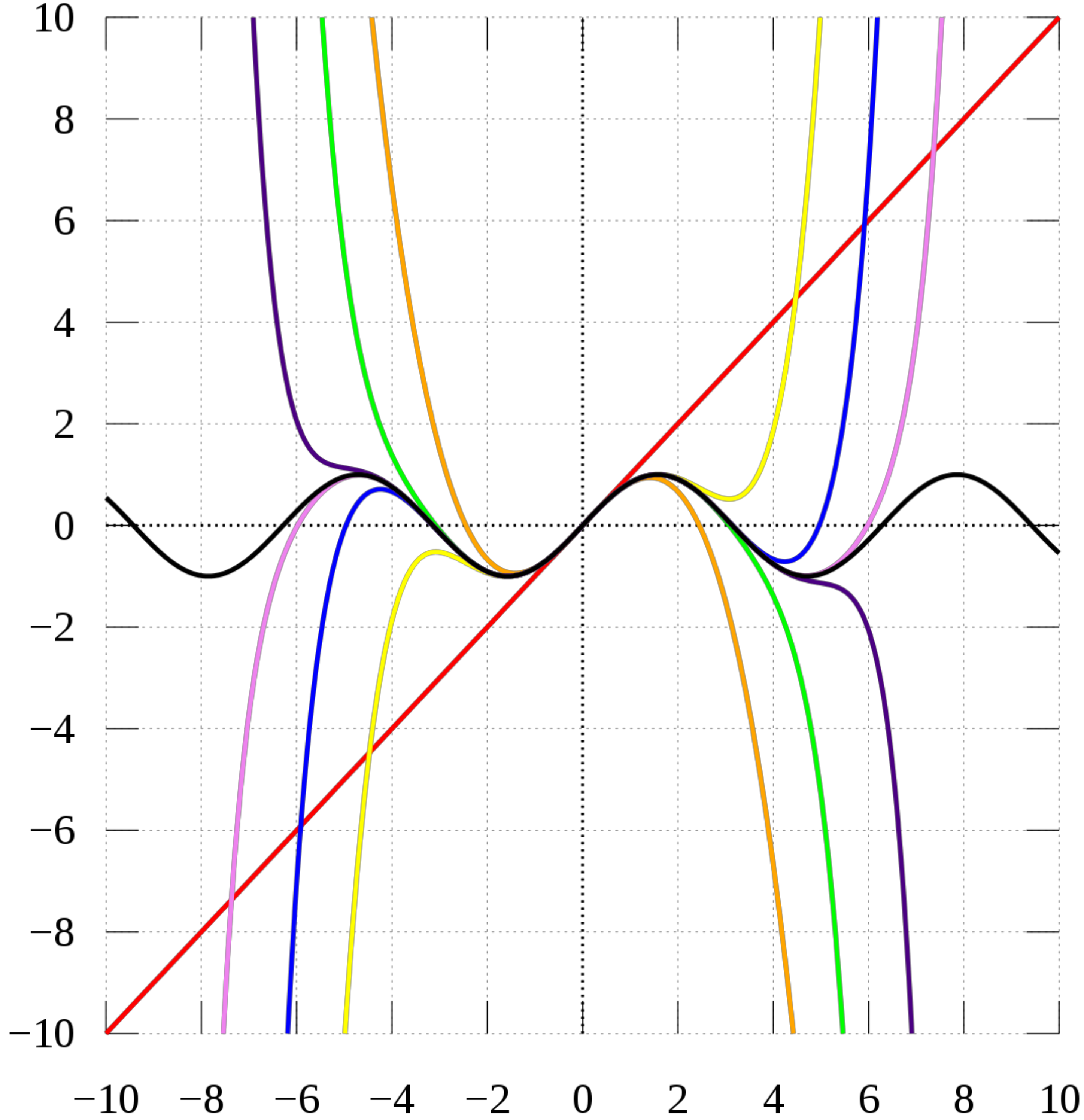
Introduction

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$$

- How to derive the series for a given function?
- How many terms should we add?

  or
- How good is our approximation if we only sum up the first N terms?

A general form of approximation is in terms of Taylor Series.



Taylor's Theorem

Taylor's Theorem: If the function f and its first $n+1$ derivatives are continuous on an interval containing a and x, then the value of the function at x is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

where the remainder R_n is defined as

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

(the integral form)

Derivative or Lagrange Form of the remainder

The remainder R_n can also be expressed as

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{(the Lagrange form)}$$

for some c between a and x

The Lagrange form of the remainder <u>makes analysis of truncation errors easier</u>.

Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

- Taylor series provides a mean to approximate any smooth function as a polynomial.

- Taylor series provides a mean to predict a function value at one point x in terms of the function and its derivatives at another point a.

- We call the series "Taylor series of f at a" or "Taylor series of f about a".

Example – Taylor Series of e^x at 0

$f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x \Rightarrow f^{(k)}(x) = e^x$ for any $k \geq 0$

Thus $f^{(k)}(0) = 1$ for any $k \geq 0$.

With $a = 0$, the Taylor series of f at 0 becomes

$$f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

Note:

Taylor series of a function f at 0 is also known as the Maclaurin series of f.

Exercise – Taylor Series of $\cos(x)$ at 0

$f(x) = \cos(x) \Rightarrow f(0) = 1$ $\quad f'(x) = -\sin(x) \Rightarrow f'(0) = 0$

$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$ $\quad f^{(3)}(x) = \sin(x) \Rightarrow f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(0) = 1$ $\quad f^{(5)}(x) = -\sin(x) \Rightarrow f^{(5)}(0) = 0$

$\vdots$

With $a = 0$, the Taylor series of f at 0 becomes

$$f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \ldots$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Question

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$$

What will happen if we sum up only the first $n+1$ terms?

Truncation Errors

Truncation errors are the errors that result from using an approximation in place of an exact mathematical procedure.

Approximation

Truncation Errors

$$e^x = \underbrace{\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}} + \boxed{\frac{x^{n+1}}{(n+1)!} + \ldots}}_{\text{Exact mathematical formulation}}$$

Exact mathematical formulation

How good is our approximation?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \boxed{\frac{x^{n+1}}{(n+1)!} + \ldots}$$

How big is the truncation error if we only sum up the first $n+1$ terms?

To answer the question, we can analyze the remainder term of the Taylor series expansion.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \boxed{R_n}$$

Analyzing the remainder term of the Taylor series expansion of $f(x)=e^x$ at 0

The remainder R_n in the Lagrange form is

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some c between a and x

For $f(x) = e^x$ and $a = 0$, we have $f^{(n+1)}(x) = e^x$. Thus

$$R_n = \frac{e^c}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ in } [0,\ x]$$

$$\leq \left| \frac{e^x}{(n+1)!}x^{n+1} \right|$$

We can estimate the largest possible truncation error through analyzing R_n.

Example

Estimate the truncation error if we calculate e as

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{7!}$$

This is the Maclaurin series of $f(x)=e^x$ with $x = 1$ and $n = 7$. Thus the bound of the truncation error is

$$R_7 \leq \left| \frac{e^x}{(7+1)!} x^{7+1} \right| = \left| \frac{e^1}{8!} (1)^8 \right| = \left| \frac{e}{8!} \right| \approx 0.6742 \times 10^{-4}$$

The actual truncation error is about 0.2786×10^{-4}.

Observation

For the same problem, with $n = 8$, the bound of the truncation error is

$$R_8 \leq \left|\frac{e}{9!}\right| \approx 0.7491 \times 10^{-5}$$

With $n = 10$, the bound of the truncation error is

$$R_{10} \leq \left|\frac{e}{11!}\right| \approx 0.6810 \times 10^{-7}$$

More terms used implies better approximation.

Example (Backward Analysis)

This is the Maclaurin series expansion for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

If we want to approximate $e^{0.01}$ with an error less than 10^{-12}, at least how many terms are needed?

With $x = 0.01, \quad 0 \le c \le 0.01, \quad f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$

$$R_n = \frac{e^c}{(n+1)!} x^{n+1} \le \frac{e^{0.01}}{(n+1)!}(0.01)^{n+1} < \frac{1.1}{(n+1)!}(0.01)^{n+1}$$

Note:1.1^{100} is about 13781 > e

To find the smallest n such that $R_n < 10^{-12}$, we can find the smallest n that satisfies

$$\frac{1.1}{(n+1)!}(0.01)^{n+1} < 10^{-12}$$

With the help of a computer:

```
n=0 Rn=1.100000e-02
n=1 Rn=5.500000e-05
n=2 Rn=1.833333e-07
n=3 Rn=4.583333e-10
n=4 Rn=9.166667e-13
```

So we need at least 5 terms

Same problem with larger step size

With $x = 0.5,\quad 0 \le c \le 0.5,\quad f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$

$$R_n = \frac{e^c}{(n+1)!}x^{n+1} \le \frac{e^{0.5}}{(n+1)!}(0.5)^{n+1} < \frac{1.7}{(n+1)!}(0.5)^{n+1}$$

Note: 1.7^2 is 2.89 > e

With the help of a computer:

```
n=0 Rn=8.500000e-01
n=1 Rn=2.125000e-01
n=2 Rn=3.541667e-02
n=3 Rn=4.427083e-03
n=4 Rn=4.427083e-04
n=5 Rn=3.689236e-05
n=6 Rn=2.635169e-06
n=7 Rn=1.646980e-07
n=8 Rn=9.149891e-09
n=9 Rn=4.574946e-10
n=10 Rn=2.079521e-11
n=11 Rn=8.664670e-13
```

So we need at least 12 terms

To approximate $e^{10.5}$ with an error less than 10^{-12}, we will need at least 55 terms. (Not very efficient)

How can we speed up the calculation?

Exercise

If we want to approximate $e^{10.5}$ with an error less than 10^{-12} using the Taylor series for $f(x)=e^x$ <u>at 10</u>, at least how many terms are needed?

The Taylor series expansion of $f(x)$ at 10 is

$$f(x) = f(10) + \frac{f'(10)}{1!}(x-10) + \frac{f''(10)}{2!}(x-10)^2 + \ldots + \frac{f^{(n)}(10)}{n!}(x-10)^n + R_n$$

$$= e^{10}\left(1 + (x-10) + \frac{(x-10)^2}{2!} + \ldots + \frac{(x-10)^n}{n!}\right) + R_n$$

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-10)^{n+1} \quad \text{for some } c \text{ between 10 and } x$$

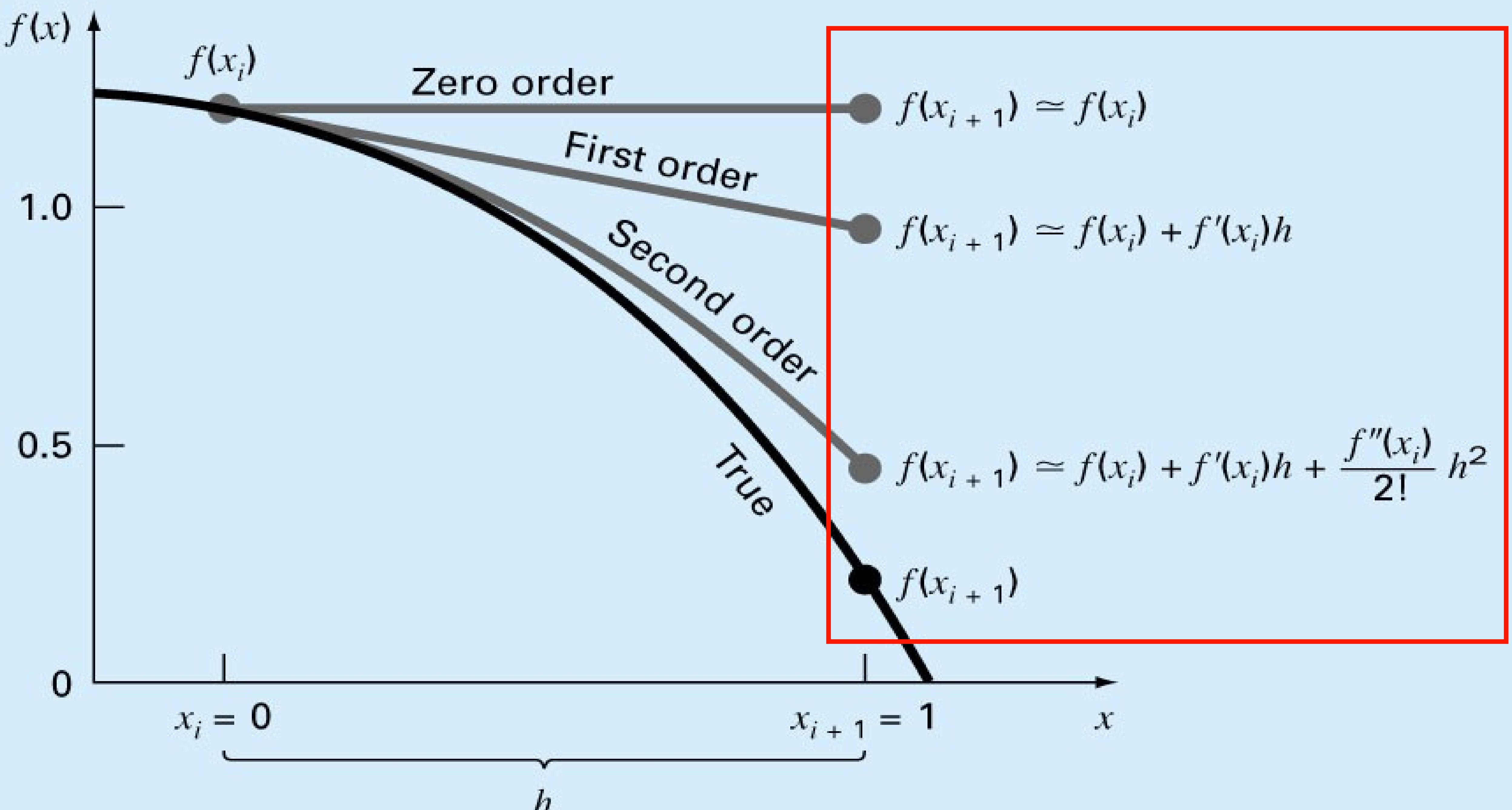
The smallest n that satisfy $R_n < 10^{-12}$ is $n = 18$. So we need at least 19 terms.

Observation

- A Taylor series converges rapidly near the point of expansion and slowly (or not at all) at more remote points.

Taylor Series Approximation Example:

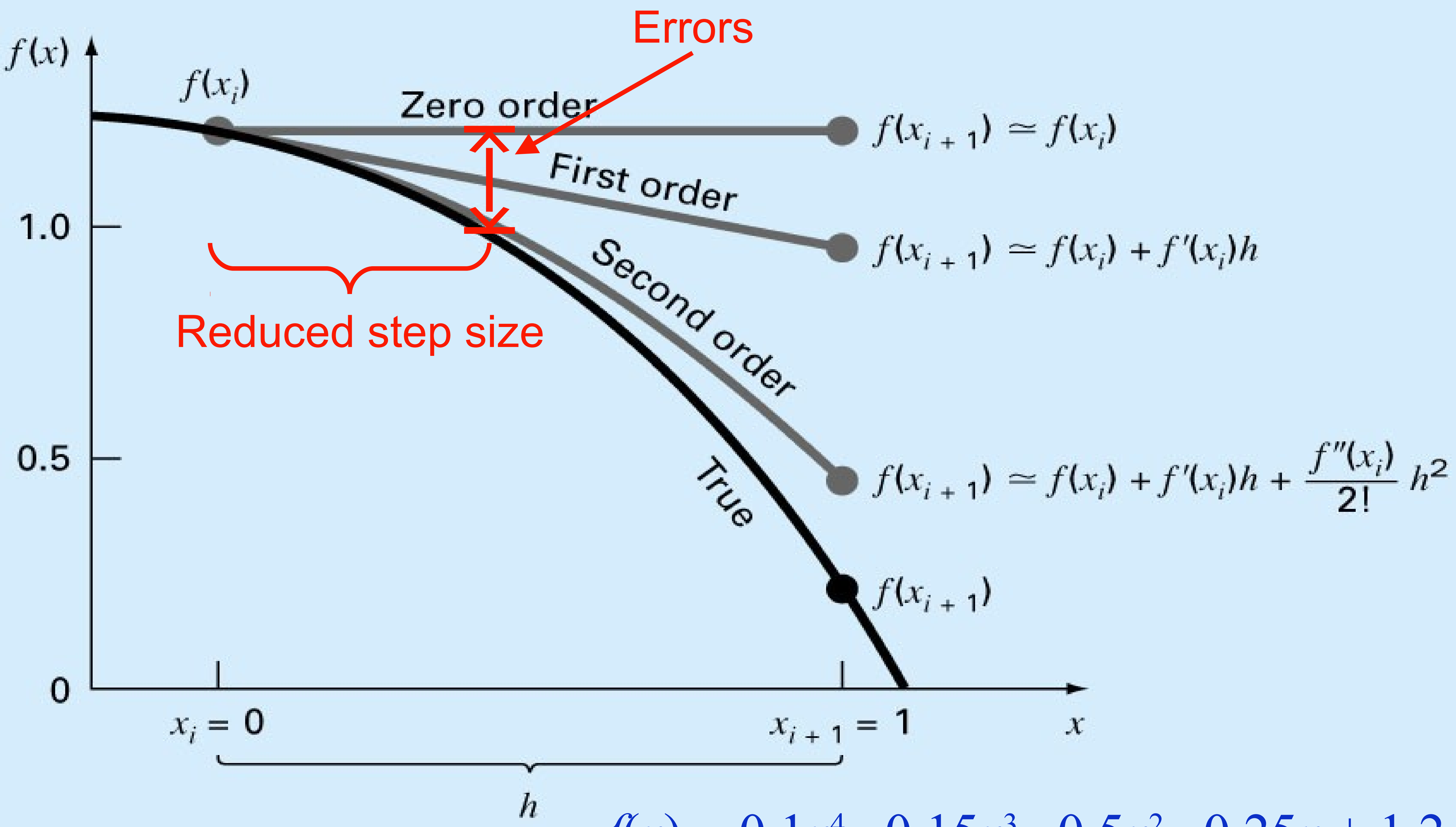
More terms used implies better approximation



$$f(x) = 0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Taylor Series Approximation Example:

Smaller step size implies smaller error



$$f(x) = 0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Taylor Series (Another Form)

If we let $h = x - a$, we can rewrite the Taylor series and the remainder as

$$f(x) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n + R_n$$

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}h^{n+1}$$

When h is small, h^{n+1} is much smaller.

h is called the step size.

h can be +ve or –ve.

The Remainder of the Taylor Series Expansion

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!} h^{n+1} = O(h^{n+1})$$

Summary

To reduce truncation errors, we can reduce h or/and increase n.

If we reduce h, the error will get smaller quicker (with less n).

This relationship has <u>no implication on the magnitude of the errors</u> because the constant term can be huge! It only give us an estimation on how much the truncation error would reduce when we reduce h or increase n.

Other methods for estimating truncation errors of a series

$$S = \underbrace{t_0 + t_1 + t_2 + t_3 + \ldots + t_n}_{S_n} + \underbrace{t_{n+1} + t_{n+2} + t_{n+3} + \ldots}_{R_n}$$

1. By Geometry Series
2. By Integration
3. Alternating Convergent Series Theorem

Note: Some Taylor series expansions may exhibit certain characteristics which would allow us to use different methods to approximate the truncation errors.

Estimation of Truncation Errors By Geometry Series

If $|t_{j+1}| \le k|t_j|$ where $0 \le \text{k} < 1$ for all $j \ge n$, then

$$
\begin{aligned}
|R_n| &= t_{n+1} + t_{n+2} + t_{n+3} + \ldots \\
&\le t_{n+1} + k|t_{n+1}| + k^2|t_{n+1}| + \ldots \\
&= |t_{n+1}|(1 + k + k^2 + k^3 + \ldots) \\
&= \frac{|t_{n+1}|}{1-k}
\end{aligned}
$$

$$|R_n| \le \frac{k|t_n|}{1-k}$$

Example (Estimation of Truncation Errors by Geometry Series)

What is $|R_6|$ for the following series expansion?

$$S = 1 + \pi^{-2} + \sqrt{2}\pi^{-4} + \sqrt{3}\pi^{-6} + \ldots + \sqrt{j}\pi^{-2j} + \ldots$$

$$t_j = \sqrt{j}\pi^{-2j}$$

Solution:

Is there a k $(0 \le k < 1)$ s.t.

$|t_{j+1}| \le k|t_j|$ or $|t_{j+1}|/|t_j| \le k$

for all $j \le n$ $(n=6)$?

If you can find this k, then

$$|R_n| \le \frac{k|t_n|}{1-k}$$

$$\frac{|t_{j+1}|}{|t_j|} = \frac{\sqrt{j+1}\pi^{-2j-2}}{\sqrt{j}\pi^{-2j}} = \sqrt{1+\frac{1}{j}}\pi^{-2}$$

$$\frac{|t_{j+1}|}{|t_j|} \le \sqrt{1+\frac{1}{6}}\pi^{-2} \quad \forall j \ge 6$$

$$< 0.11$$

$$k = 0.11, \quad |t_6| < 3 \times 10^{-6}$$

$$|R_6| \le \frac{k|t_n|}{1-k} < \frac{0.11}{1-0.11} \times 3 \times 10^{-6}$$

Estimation of Truncation Errors By Integration

If we can find a function $f(x)$ s.t. $|t_j| \le f(j)\ \forall j \ge n$

and $f(x)$ is a decreasing function $\forall x \ge n$, then

$$|R_n| = t_{n+1} + t_{n+2} + t_{n+3} + \ldots = \sum_{j=n+1}^{\infty} |t_j| \le \sum_{j=n+1}^{\infty} f(j)$$

$$|R_n| \le \int_n^{\infty} f(x)\,dx$$

Example (Estimation of Truncation Errors by Integration)

Estimate $|R_n|$ for the following series expansion.

$$S = \sum_{j=1} t_j \quad \text{where} \quad t_j = (j^3 + 1)^{-1}$$

Solution:

We can pick $f(x) = x^{-3}$ because it would provide a tight bound for $|t_j|$. That is

$$\frac{1}{j^3} \geq \frac{1}{1 + j^3} \quad \forall j \geq 1$$

So $$\left| R_n \right| \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

Alternating Convergent Series Theorem (Leibnitz Theorem)

If an infinite series satisfies the conditions

- It is strictly alternating.
- Each term is smaller in magnitude than that term before it.
- The terms approach to 0 as a limit.

Then the series has a finite sum (i.e., converge) and moreover if we stop adding the terms after the n^{th} term, the error thus produced is between 0 and the 1^{st} non-zero neglected term not taken.

Alternating Convergent Series Theorem

Example 1:

Maclaurin series of $\ln(1+x)$

$$S = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad (-1 < x \le 1)$$

With $n = 5$,

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.7833333340$$

$$\ln 2 = 0.693$$

$$|R| = |S - \ln 2| \approx 0.09 \le \frac{1}{6} = 0.16666$$

Actual error

Eerror estimated using the althernating convergent series theorem

Alternating Convergent Series Theorem

Example 2:

Maclaurin series of $\cos(x)$

$$S = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

With $n = 5$,

$$S = 1 - \frac{1^2}{2!} + \frac{1^4}{4!} - \frac{1^6}{6!} + \frac{1^8}{8!} = 0.5403025793$$

$$\cos(1) = 0.5403023059$$

$$\left|S - \cos(1)\right| = 2.73 \times 10^{-7} \leq \frac{1}{10!} = 2.76 \times 10^{-7}$$

Actual error

Eerror estimated using the althernating convergent series theorem

Exercise

If the sine series is to be used to compute $\sin(1)$ with an error less than $0.5\text{x}10^{-14}$, how many terms are needed?

$$\sin(1) = 1 - \frac{1^3}{3!} + \frac{1^5}{5!} - \frac{1^7}{7!} + \frac{1^9}{9!} - \frac{1^{11}}{11!} + \frac{1^{13}}{13!} - \frac{1^{15}}{15!} + \frac{1^{17}}{17!} - \ldots$$

$R_0 \quad R_1 \quad R_2 \quad R_3 \quad R_4 \quad R_5 \quad R_6 \quad R_7$

Solution:

This series satisfies the conditions of the Alternating Convergent Series Theorem.

Solving $$R_n \leq \left| \frac{1}{(2n+3)!} \right| \leq \frac{1}{2} \times 10^{-14}$$

for the smallest n yield $n = 7$ (We need 8 terms)

Exercise

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$$

How many terms should be taken in order to compute $\pi^4/90$ with an error of at most 0.5×10^{-8}?

$$|R_n| = t_{n+1} + t_{n+2} + \ldots = \sum_{j=n+1}^{\infty} \frac{1}{(j+1)^4} < \int_n^{\infty} (x+1)^{-4}\,dx$$

Solution (by integration):

$$= \left.\frac{(x+1)^{-3}}{-3}\right|_n^{\infty} = \frac{(n+1)^{-3}}{3}$$

$$\left|\frac{1}{3(n+1)^3}\right| < \frac{1}{2} \times 10^{-8} \Rightarrow (n+1) \geq 406 \Rightarrow n \geq 405$$

Note: If we use $f(x) = x^{-3}$ (which is easier to analyze) instead of $f(x) = (x+1)^{-3}$ to bound the error, we will get n >= 406 (just one more term).

Summary

- Understand what truncation errors are
- Taylor's Series
  - Derive Taylor's series for a "smooth" function
  - Understand the characteristics of Taylor's Series approximation
  - Estimate truncation errors using the remainder term
- Estimating truncation errors using other methods
  - Alternating Series, Geometry series, Integration